Diagnostic Test for Incoming Graduate Students

If you struggle with the majority of these questions, you will have to work particularly hard to succeed in core classes. You should plan your schedule accordingly, so that you have the time and bandwidth to do so.

1 Quantum Mechanics

Sakurai Chapter 1 Problem 1: Commutator Algebra

Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB. \tag{1}$$

('$[\cdot,\cdot]$' indicates commutator, '$\{\cdot,\cdot\}$' is anticommutator, and A, B, C, and D are all operators).

Sakurai Chapter 1 Problem 4: Bra-ket Algebra

Using the rules of bra-ket algebra, prove or evaluate the following:

(a) $tr(XY) = tr(YX)$ where X and Y are operators.

(b) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.

(c) $\exp(i * f(A)) =?$ in ket-bra form, where A is a Hermitian operator whose eigenalues are known (denote the eigenvalues as a_i and the eigenkets as $|a_i\rangle$).

(d) $\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'')$, where $\psi_{a'}(\mathbf{x}') = \langle \mathbf{x}'|a'\rangle$.

Sakurai Chapter 1 Problem 10: Diagonalizing Hamiltonians

The Hamiltonian operator for a two-state system is given by

$$H = a\big(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \big), \tag{2}$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

Sakurai Chapter 1 Problem 29: Commutation Relations with Functions

(a) On page 247, Gottfried (1966) states that

$$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \quad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i} \tag{3}$$

can be "easily derived" from the fundamental commutation relations for all functions F and G that can be expressed as power series in their arguments. Verify this statement.

(b) Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classical}}$.

Griffiths 2.14: Simple Harmonic Oscillator

A particle is in the ground state of the harmonic oscillator with classical freuqency ω, when suddenly the spring constant quadruples, so $\omega' = 2\omega$, without initially changing the wave function (of course, Ψ will now *evolve* differently, because the Hamiltonian has changed). What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$? What is the probability of getting $\hbar\omega$?

Griffiths Problem 7.1: Variational Method

Use a Gaussian trial function, namely

$$\psi(x) = Ae^{-bx^2}, \tag{4}$$

to obtain the lowest upper bound you can on the ground state energy of

(a) the linear potential: $V(x) = \alpha\,|x|$

(b) the quartic potential: $V(x) = \alpha x^4$

Zetilli Problem 5.5: Spin Eigenstates

(a) Find the eigenvalues and eigenstates of the spin operator $\vec{S}$ of an electron in the direction of a unit vector $\vec{n}$, where $\vec{n}$ is *arbitrary.*

(b) Find the probability of measuring $\hat{S}_z = -\hbar/2$

(c) Assuming that the eigenvectors of the spin calculated in (a) correspond to $t = 0$, find these eigenvectors at time t.

2 Electromagnetism

Jackson Problem 1.1: Basics

Use Gauss's theorem (and $\oint \mathbf{E}\cdot d\mathbf{l} = 0$ if necessary) to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface (A conductor by definition contains charges capable of moving freely under the action of applied electric fields).

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.

Jackson Problem 2.1: Method of Images

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Using the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

(c) the total force acting on the plane by integrating $\sigma^2/2\epsilon_0$ over the whole plane;

(d) the work necessary to remove the charge q from its position to infinity;

(e) the potential energy between the charge q and its image (compare the answer to part (d) and discuss).

(f) Find the answer to part (d) in electron volts for an electron originally one angstrom from the surface.

Jackson 5.33: Mutual Inductance

Consider two current loops (as in the figure below) whose orientation in space is fixed, but whose relative separation can be changed. Let O_1 and O_2 be origins in the two loops, fixed relative to each loop, and $\mathbf{x}_1$ and $\mathbf{x}_2$ be coordinates of elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$, respectively, of the loops referred to the respective origins. Let $\mathbf{R}$ be the relative coordinate of the origins, directed from loop 2 to loop 1.

(a) Starting from

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{(d\mathbf{l}_1 \cdot d\mathbf{l}_2)\mathbf{x}_{12}}{|\mathbf{x}_{12}|^3}, \tag{5}$$

the expression for the force between the loops, show that it can be written

$$\mathbf{F}_{12} = I_1 I_2 \nabla_R M_{12}(\mathbf{R}) \tag{6}$$

where M_{12} is the mutual inductance of the loops,

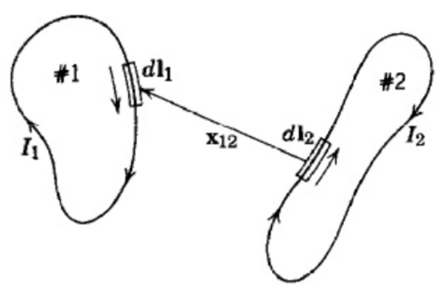
$$M_{12}(\mathbf{R}) = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{R}|}, \tag{7}$$

and it is assumed that the orientation of the loops does not change with $\mathbf{R}$

(b) Show that the mutual inductance, viewed as a function of $\mathbf{R}$, is a solution of the Laplace equation,

$$\nabla_R^2 M_{12}(\mathbf{R}) = 0 \tag{8}$$

The importance of this result is that the uniqueness of solutions of the Laplace equation allows the exploitation of the properties of such solutions, provided a solution can be found for a particular value of $\mathbf{R}$.



Griffiths 5.26: Magnetic Vector Potentials

(a) By whatever means you can think of (short of looking it up), find the vector potential a distance s from an infinite straight wire carrying a current I. Check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

(b) Find the magnetic potential *inside* the wire, if it has radius R and the current is uniformly distributed.

Griffiths 7.37: Maxwell's Equations

Suppose

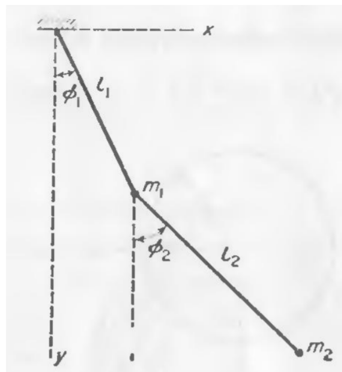
$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \theta(r - vt)\hat{\mathbf{r}}; \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{0}. \tag{9}$$

Show that these fields satisfy all of Maxwells equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

3 Classical Mechanics

Landau and Lifshitz Chapter 1 Problem 1: Double Pendulum

Find the Lagrangian for a coplanar double pendulum (see below) when placed in a uniform gravitational field (acceleration g).



Landau and Lifshitz Chapter 2 Problem 1: Conservation of Momentum

A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which its potential energy is constant U_1 and enters another in which its potential energy is a different constant U_2. Determine the change in the direction of motion of the particle.

Taylor Problem 5.40: Resonance

Consider a damped oscillator, with fixed natural frequency ω_0 and fixed damping constant β (not too large), that is driven by a sinusoidal force with variable frequency ω. Show that the amplitude of the response, as given by

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{10}$$

is a maximum when $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. (Note that so long as the resonance is narrow this implies $\omega \approx \omega_0$).

Taylor Problem 11.4: Normal Modes of Coupled Systems

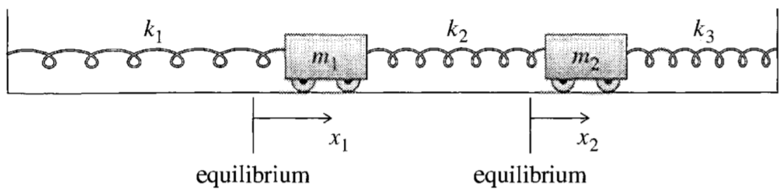
(a) Find the normal frequencies for the system of two carts and three springs shown in the figure below, for the case that $m_1 = m_2$ and $k_1 = k_3$ (but k_2 may be different). Check that your answer is correct for the case that $k_1 = k_2$ as well

(b) Find and describe the motion in each of the two normal modes in turn. Compare with the motion found for the case that $k_1 = k_2$ in Section 11.2. Explain any similarities.

[Note: the normal modes when $k_1 = k_2$ are given by:

1. $x_1(t) = A\cos\Big(\sqrt{3k/m}t - \delta\Big)$; $x_2(t) = -A\cos\Big(\sqrt{3k/m}t - \delta\Big)$
2. $x_1(t) = A\cos\Big(\sqrt{k/m}t - \delta\Big)$; $x_2(t) = -A\cos\Big(\sqrt{k/m}t - \delta\Big)$

]



Goldstein 3.19: Orbital Motion

A particle moves in a force field described by

$$F(r) = -\frac{k}{r^2}\exp\left(\frac{-r}{a}\right) \tag{11}$$

where k and a are positive.

(a) Write the equations of motion and reduce them to the equivalent one-dimensional problem. Use the effective potential to discuss the qualitative nature of the orbits for different values of the energy and the angular momentum.

(b) Show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\rho/a$ per revolutions, where ρ is the radius of the circular orbit.

Goldstein 8.15: Hamiltonians and Lagrangians

A dynamical system has the Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2, \tag{12}$$

where a, b, k_1, and k_2 are constants. Find the equations of motion in the Hamiltonian formulation.

4 Statistical Mechanics

Schroeder Problem 3.10: Entropy and Heat

An ice cube (mass 30 g) at 0°C is left sitting on the kitchen table, where it gradually melts. The temperature in the kithen is 25°C.

(a) Calculate the change in the entropy of the ice cube as it melts into water at 0°C (Don't worry about the fact that the volume changes somewhat).

(b) Calculate the change in the entropy of the water (from the melted ice) as it temperature rises from 0°C to 25°C.

(c) Calculate the change in the entropy of the kitchen as it gives up heat to the melting ice/water.

(d) Calculate the net change in the entropy of the universe during this process. Is the net change positive, negative, or zero? Is this what you would expect?

Schroeder 5.12: Maxwell Relations

Functions encountered in physics are generally well enough behaved that their mixed partial derivatives do not depend on which derivative is taken first. Therefore, for instance,

$$\frac{\partial}{\partial V}\left(\frac{\partial U}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial U}{\partial V}\right), \tag{13}$$

where each $\partial/\partial V$ is taken with S fixed, each $\partial/\partial S$ is taken with V fixed, and N is always held fixed. From the thermodynamic identity (for U) you can evaluate the parital derivatives in parentheses to obtain

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V, \tag{14}$$

a nontrivial identity called a **Maxwell relation**. Go through the derivation of this relation step by step. Then derive an analogous Maxwell relation for each of the other three thermodynamic identities discussed in the text (for H, F, and G). Hold N fixed in all the partial derviatives; other Maxwell relations *can* be derived by considering partial derivatives with respect to N, but after you've done four of them the novelty begins to wear off.

Schroeder Problem 6.45: Derive Everything from the Partition Function

Derive the following equations:

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = Nk\left[\ln\left(\frac{V}{Nv_Q}\right) + \frac{5}{2}\right] - \frac{\partial F_{int}}{\partial T} \tag{15}$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\ln\left(\frac{VZ_{int}}{Nv_Q}\right) \tag{16}$$

for the entropy and chemical potential of an ideal gas.

Schroeder Problem 7.28: Fermi Gas and Density of States

Consider a free Fermi gas in two dimensions, confined to a square area $A = L^2$.

(a) Find the Fermi energy (in terms of N and A), and show that the average energy of the particles is $\epsilon_F/2$.

(b) Derive a formula for the density of states. You should find that is is a constant, independent of ϵ.

(c) Explain how the chemical potential of this system should behave as a function of temperature, both when $kT \ll \epsilon_F$ and when T is much higher.

(d) Because $g(\epsilon)$ is a constant for this state, it is possible to carry out the integral

$$N = \int_0^\infty g(\epsilon)\bar{n}_{FD}(\epsilon)d\epsilon = \int_0^\infty g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}+1}d\epsilon \tag{17}$$

for the number of particles analytically. Do so, and solve for μ as a function of N. Show that the resulting formula has the expected qualitative bahavior.

(e) Show that in the high-temperature limit, $kT \gg \epsilon_F$, the chemical potential of this system is the same as that of an ordinary ideal gas.

Schroeder Problem 7.72: Bose Einstein Condensation

For a gas of particles confined inside a two-dimensional box, the density of states is constant, independent of ϵ (see previous problem). Investigate the behavior of a gas of noninteracting bosons in a two-dimensional box. You should find that the chemical potential remains significantly less than zero as long as T is significantly greater than zero, and hence that there is no abrupt condensation of particles into the ground state. Explain how you know that this is the case, and describe what *does* happen to this system as the temperature decreases. What property must $g(\epsilon)$ have in order for there to be an abrupt Bose-Einstein condensation?