Entanglement Entropy, QFT and Holography

Or what some string theorists do nowadays?

Julio Virrueta

Stony Brook University

April 5, 2017

Outline

1. Introduction
2. Entanglement in QM
3. Entanglement in QFT
4. Entanglement in String Theory

Introduction

Since it's first proposal, Entanglement Entropy has been a topic of interest in several areas of physics:

- Validity test for quantum mechanics: Bell inequalities.
- Many-body quantum mechanics: Tensor networks and phase transitions.
- Quantum Information and Computing: Practically the whole field.
- QFT and String Theory

Entanglement in QM

Basics of QM

States and Hilbert space:

$$|\psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle \in \mathcal{H}$$

Probabilities:

$$P(\uparrow) = |\alpha|^2$$
$$P(\downarrow) = |\beta|^2$$

(Einstein: this is bullshit...hidden variables)

Multi-Body QM

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\psi\rangle_{AB} = \sum_{ij} \omega_{ij} \, |\phi_i\rangle_A \otimes |\chi_j\rangle_B$$

For instance:

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|\uparrow, \downarrow\rangle \pm |\downarrow, \uparrow\rangle \right)$$

EPR paradox

If we measure the state on A and observe it to be in the state $|\uparrow\rangle$ the state on B will be, with probability 1, in the state $|\downarrow\rangle$, $|\psi\rangle_{AB}$ is said to be *entangled*.

Entangled state are such that cannot be written as:

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

(Einstein: You see? Bullshit)

Bell's theorem

No physical theory of local hidden variables can ever reproduce all of the predictions of quantum mechanics.

Entanglement is the quintessential QM phenomena!

(Eintein's attemp to show QM is wrong... show QM is the only choice)

Density Matrix and Mix States

For state $|\psi\rangle$:

$$\rho = |\psi\rangle \langle\psi|$$

A generalization, *mixed states*:

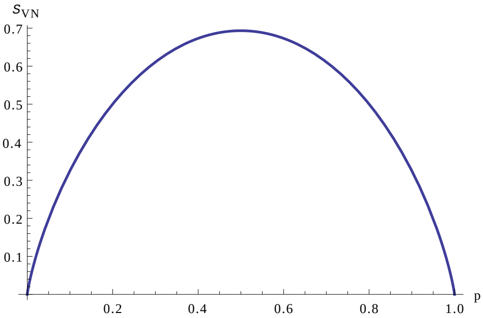
$$\rho = \sum_i p_i \rho_i = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

How mixed is a state?

$$S_{VN} = -\mathrm{Tr}\rho \log \rho = -\sum_i p_i \log p_i$$

Example: binary state with $p_1 = p$ and $p_2 = 1 - p$:

$$S_{VN} = -p \log p - (1-p) \log(1-p)$$



Entanglement Entropy

Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho = |\psi\rangle \langle\psi|$

Define the *reduced density matrix*:

$$\rho_A = \mathsf{Tr}_B \rho$$

$$|\psi\rangle = |\phi_A\rangle \otimes |\chi_B\rangle \Leftrightarrow \rho_A = |\phi_A\rangle \langle\phi_A|$$

Entangled $\Leftrightarrow$ Mixed

Define the *Entanglement Entropy* as:

$$S(A) = -\text{Tr}\rho_A \log \rho_A$$

Some properties:

- $S(A) = S(-A)$
- Strong Subadditivity:

$$S(ABC) + S(B) \leq S(AB) + S(BC)$$

If you like this and want to know more about it:

- http://www.theory.caltech.edu/people/preskill/ph229/
- Steven Weinberg. Lectures on quantum mechanics. 2013.
- A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? Phys. Rev., 47:777–780, May 1935.

Intermedio

(QFT and String Theory to follow)

Entanglement in QFT

(Con dibujitos)

Density matrix in QFT

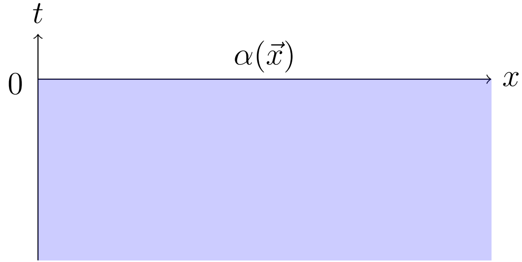
Real scalar field theory, $\hat{\phi}(\vec{x}, t)$.

The Hilbert space is spanned by $\{|\alpha(x)\rangle\}$

$$\hat{\phi}(x)\,|\alpha(x)\rangle = \alpha(x)\,|\alpha(x)\rangle$$

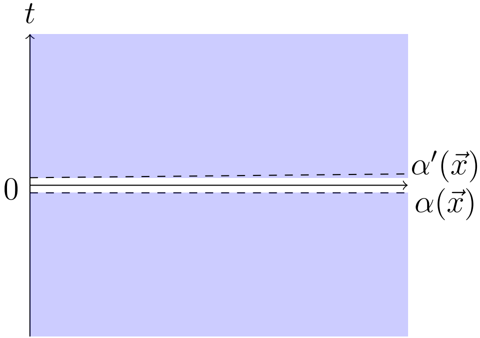
Ground state functional:

$$\Phi(\alpha) = \langle 0|\alpha\rangle \sim \int_{\phi(\vec{x},-\infty)=0}^{\phi(\vec{x},0)=\alpha(\vec{x})} D\phi(x) e^{-S_E[\phi]}$$



Ground state density matrix: $\rho = |0\rangle\langle 0|$ and matrix elements:

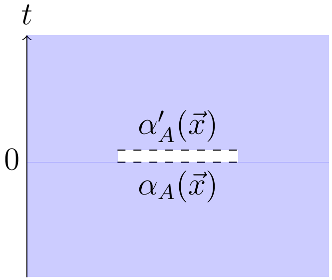
$$\rho(\alpha', \alpha) = \langle \alpha' | 0 \rangle \langle 0 | \alpha \rangle = \Phi(\alpha')^* \Phi(\alpha)$$



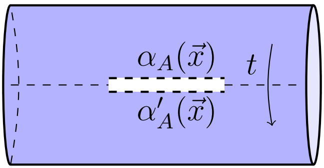
For $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, $\alpha = \alpha_A \times \beta_{\bar{A}}$:

$$\rho_A(\alpha'_A, \alpha_A) = \mathrm{Tr}_B \rho \sim \int D\beta \Phi(\alpha')^* \Phi(\alpha)$$

$$\sim \int_{\phi(\vec{x},0^-)=\alpha_A(\vec{x});\vec{x}\in A}^{\phi(\vec{x},0^+)=\alpha'_A(\vec{x});\vec{x}\in A} D\phi(x) e^{-S_E[\phi]}$$

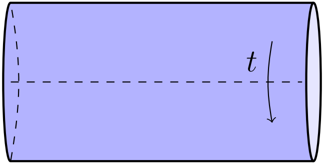


With compact time:



The trace is:

$$\mathrm{Tr}\rho_A \sim \int D\alpha \rho_A(\alpha,\alpha) \sim \int D\phi e^{-S_E[\phi]}$$



Entanglement and Rényi entropies:

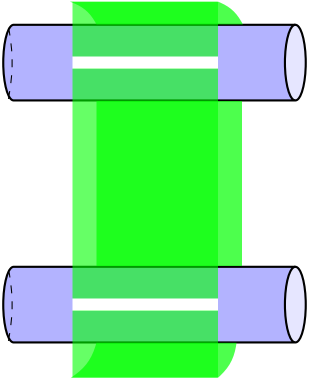
Evaluating $S(A) = -\text{Tr}\rho_A \log \rho_A$ is well... a pain in the ass.

Instead:

$$S_n(A) = \frac{1}{1-n} \log \text{Tr}\rho_A^n$$

$$\lim_{n\to 1} S_n(A) = S(A)$$

$$\mathrm{Tr}\rho_A^n \sim \int (D\alpha_1\alpha_2...\alpha_n)\, \rho_A(\alpha_1,\alpha_2)...\rho_A(\alpha_n,\alpha_1)$$
$$= \frac{Z(n)}{Z(1)^n}$$



Entanglement entropy:

$$\boxed{S(A) = \lim_{n\to 1} \frac{1}{1-n} \log \frac{Z(n)}{Z(1)^n}} \tag{1}$$

If you liked EE in QFT:

- Pasquale Calabrese and John L. Cardy. Entanglement entropy and quantum field theory. J.Stat.Mech., 0406:P06002, 2004.
- Pasquale Calabrese and John Cardy. Entanglement entropy and conformal field theory. J.Phys., A42:504005, 2009.
- H. Casini and M. Huerta. Entanglement entropy in free quantum field theory. J.Phys., A42:504007, 2009.

Entanglement in String Theory

(Más dibujitos...)

AdS/CFT correspondence

Striking String Theory result:

(Quantum) Gravity in $d+1 =$ Super Yang-Mills in d

String Theory	QFT
Type IIB Strings in $AdS_5 \times \mathbb{S}^5$	$\mathcal{N} = 4$ SYM with $SU(N)$
AdS isometries $SO(4,2)$	Conformal symmetry $SO(4,2)$
$\mathbb{S}^5$ isometries $SO(6)$	R-symmetry $SU(4)$
32 Killing spinors	32 super-charges of $\mathcal{N} = 4$
$\left(\frac{L}{l_s}\right)^4$	$\lambda = N g_{YM}^2$

AdS geometry

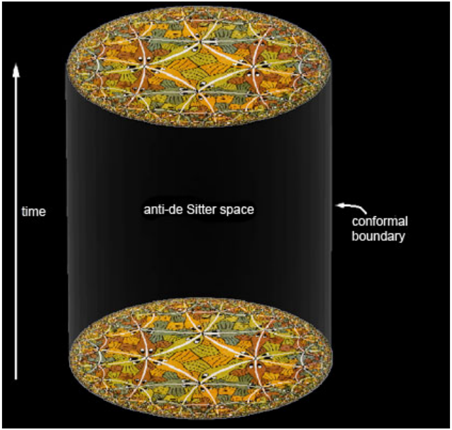
Locus:

$$-X_{-1}^2 - X_0^2 + \vec{X}^2 = -L^2$$

Coordinate patches:

$$ds^2 = \frac{L^4}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right)$$

$$ds^2 = -\left(\frac{r^2}{L^2} + 1\right)dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1} + r^2 d\Omega^2$$


time
anti-de Sitter space
conformal
boundary

Boundary conditions and sources

Field equations:

$$\left(-\Box + m^2\right)\phi(z,x) = 0$$
$$m^2 = \Delta(\Delta - d)$$

Boundary conditions:

$$\lim_{z\to 0}\phi(z,x) = z^{\Delta}\phi_0(x)$$

Correlation functions:

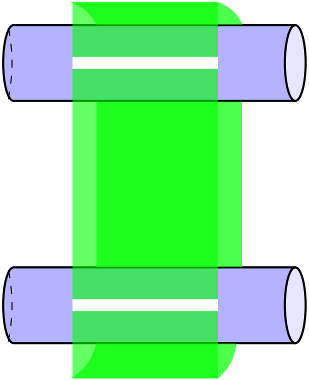
$$\boxed{\langle e^{\int d^dx\phi_0(x)\mathcal{O}_\Delta(x)}\rangle_{CFT} = Z_{SUGRA}[\phi_0]}$$

Holographic Entanglement

Recall

$$S(A) = \lim_{n \to 1} \frac{1}{1-n} \log \frac{Z(n)}{Z(1)^n} \tag{2}$$

$$\frac{Z(n)}{Z(1)^n} =$$



Vacuum CFT $\Rightarrow$ AdS geometry $\Rightarrow R = \frac{-d(d+1)}{L^2}$

Green surface $\Rightarrow 2\pi(n-1)$ angle deficit $\Rightarrow R = 4\pi(1-n)\delta(\gamma_A)$

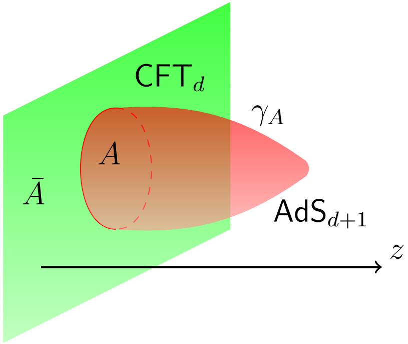
Gravitational action:

$$S = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{g}\,(R+\Lambda) = -\frac{1-n}{4G_N}\int_{\gamma_A} d^d x\sqrt{g}\ + \ldots$$
$$= -\frac{1-n}{4G_N}\text{Area}(\gamma_A)\ + \ldots$$

$$\frac{Z(n)}{Z(1)^n} = Z_{SUGRA}[\gamma_A] = \exp\left(\frac{1-n}{4G_N}\text{Area}(\gamma_A)\right)$$

$$\Rightarrow S(A) = \lim_{n\to 1}\frac{1}{1-n}\frac{1-n}{4G_N}\text{Area}(\gamma_A)$$

$$S(A) = \frac{1}{4G_N}\text{Area}(\gamma_A)$$



If you remember something from this talk, let's that be:

$$\text{Entanglement} \Leftrightarrow \text{Minimal Surface}$$

Or even better:

$$\boxed{\text{QFT Information} \Leftrightarrow \text{Geometrical information}}$$

Current problem and research opportunities:

- EE for dynamical spacetime (time dependent entanglement)
- Higher Curvature corrections
- EE for gauge and chiral theories
- Beyond EE: Geometrization of QFT (Kinematic Space)
- AdS/MERA

If you like...

AdS/CFT

- O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz. Large N Field Theories, String Theory and Gravity. Phys.Rept.323:183-386,2000
- Edward Witten. Anti-de Sitter space and holography. Adv.Theor.Math.Phys., 2:253–291, 1998.
- Eric D'Hoker, Daniel Z. Freedman. Supersymmetric Gauge Theories and the AdS/CFT Correspondence. UCLA/02/TEP/3, MIT-CTP-3242

Or Holographic EE

- Shinsei Ryu and Tadashi Takayanagi. Aspects of Holographic Entanglement Entropy. JHEP, 0608:045, 2006.
- Veronika E. Hubeny. Extremal surfaces as bulk probes in AdS/CFT. JHEP, 1207:093, 2012
- Matthew Headrick and Tadashi Takayanagi. A Holographic proof of the strong subadditivity of entanglement entropy. 10.1103/PhysRevD.76.106013

Gracias

Note: When the speaker is a overly large mexican, courtesy demands to offer him a beer.